

THE IDEAL OF CURVES OF GENUS 2 ON RATIONAL NORMAL SCROLLS

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ABSTRACT. Given a smooth curve of genus 2 embedded in \mathbf{P}^{d-2} with a complete linear system of degree $d \geq 6$, we list all types of rational normal scrolls arising from linear systems g_2^1 and g_3^1 on C .

Furthermore, we give a description of the ideal of such a curve of genus 2 embedded in \mathbf{P}^{d-2} as a sum of the ideal of the two-dimensional scroll defined by the unique g_2^1 on C and the ideal of a three-dimensional scroll arising from a g_3^1 on C and not containing the scroll defined by the g_2^1 on C .

1. INTRODUCTION

Given a projective variety X , in order to find a description of its ideal I_X which is, in some sense, easy to handle, one useful approach is to look for a decomposition of I_X as a sum of ideals of higher-dimensional varieties that contain X . This problem at hand extends naturally to the question whether the syzygies of a given variety are generated by the syzygies of higher-dimensional varieties containing this variety.

In the past decades this question has been studied for curves $C \subseteq \mathbf{P}^n$. Since each g_k^1 on C for $k \leq n-1$ gives rise to a rational normal scroll that contains the curve, natural candidates for these higher-dimensional varieties are exactly rational normal scrolls. The above presented problem has been studied to some extent for elliptic normal curves and canonical curves:

In 1984 Green ([2]) showed that the space of quadrics in the ideal of a non-hyperelliptic canonical curve of genus $g \geq 5$ is spanned by quadrics of rank less or equal to 4.

In [9] v. Bothmer and Hulek prove that the linear syzygies of elliptic normal curves, of their secant varieties and of bielliptic canonical curves are generated by syzygies of rational normal scrolls that contain these varieties.

In [7] v. Bothmer proves that the j th syzygies of a general canonical curve of genus g are generated by the j th syzygies of rational normal scrolls containing the curve for the cases $(g, j) \in \{(6, 1), (7, 1), (8, 2)\}$, and in [8] the author shows that the first syzygies of general canonical curves of genus $g \geq 9$ are generated by scrollar syzygies.

In this paper we will study curves of genus 2, which cannot be canonically embedded and are all hyperelliptic, and we will focus on the 0th syzygies, i.e. the ideal I_C .

Let C be a curve of genus 2, embedded as a smooth and irreducible curve in \mathbf{P}^{d-2} with a complete linear system of degree $d \geq 6$. Since the linear system is complete, the embedded

curve $C \subseteq \mathbf{P}^{d-2}$ is linearly normal.

Our main result is the following:

Theorem 5.1. *Let C be a smooth and irreducible curve of genus 2, linearly normal embedded in \mathbf{P}^{d-2} with a complete linear system $|H_C|$ of degree $d \geq 6$. Moreover, let S be the $g_2^1(C)$ -scroll, and let V be a $g_3^1(C)$ -scroll that does not contain S . Then we have*

$$I_S + I_V = I_C.$$

In Section 5 we will give a proof of this theorem which is based on an inductive argument. In Sections 3 and 4 we will list all possible scroll types of the unique $g_2^1(C)$ -scroll S and $g_3^1(C)$ -scrolls $V_{|D|}$. We give a connection between the linear system $|H_C|$ that embeds the curve into projective space and the scroll types of S and a $V_{|D|}$ and provide thus a proof of existence in all cases.

2. PRELIMINARIES

Let C be a non-singular curve of genus 2, and let $|H_C|$ be a complete linear system of degree $d \geq 6$ on C . By the Riemann-Roch theorem for curves (see e.g. [3], Thm. 1.3 in Chapter IV.1) the system $|H_C|$ embeds C into projective space \mathbf{P}^{d-2} . Since $|H_C|$ is complete, the embedded curve $C \subseteq \mathbf{P}^{d-2}$ is linearly normal. Throughout the whole article C will always denote a smooth curve of degree $d \geq 6$ in \mathbf{P}^{d-2} .

We use the notation $g_k^1(C)$ to denote a g_k^1 on C . We are interested in rational normal scrolls that arise from the unique $g_2^1(C)$ and from $g_3^1(C)$'s. The following proposition states that indeed there is only one g_2^1 on C , and furthermore it computes the dimension of the family of g_3^1 's on C , which we will denote by $G_3^1(C)$:

Proposition 2.1. *There exists exactly one $g_2^1(C)$, and this is equal to the canonical system $|K_C|$. The family $G_3^1(C) := \{g_3^1(C)'s\}$ is two-dimensional.*

Proof. We use the Riemann-Roch theorem for curves (see e.g. [3], Thm. 1.3 in Chapter IV.1):

If D is a divisor of degree 2, then

$$h^0(\mathcal{O}_C(D)) = 1 + h^0(\mathcal{O}_C(K_C - D)) = \begin{cases} 1 & \text{if } D \notin |K_C|, \\ 2 & \text{if } D \in |K_C|. \end{cases}$$

Hence we can conclude that the linear system $|D|$ is a $g_2^1(C)$ if and only if $|D| = |K_C|$.

If D is a divisor of degree 3 on C , then, again by the Riemann-Roch theorem for curves, $h^0(\mathcal{O}_C(D)) = 2$, i.e. each linear system $|D|$ of degree 3 is a $g_3^1(C)$. The set of all effective divisors of degree 3 on C is given by $C_3 := (C \times C \times C)/S_3$, where S_3 denotes the symmetric group on 3 letters. The dimension of this family is equal to 3, and since each linear system $|D|$ of degree 3 has dimension 1, as shown above, the family of $g_3^1(C)$'s has to be two-dimensional. \square

We are interested in the rational normal scroll arising from the unique $g_2^1(C)$, and for each $|D|$ in the two-dimensional family $G_3^1(C)$ we are interested in the scroll defined by $|D|$. There are several different presentations of a rational normal scroll. We will use two of these, which will be given in the following paragraphs.

Definition 2.2. (cf. [5])

Let e_1, e_2, \dots, e_k be integers with $e_1 \geq e_2 \geq \dots \geq e_k \geq 0$ and $e_1 + e_2 + \dots + e_k \geq 2$. Set $\mathcal{E} = \mathcal{O}_{\mathbf{P}^1}(e_1) \oplus \mathcal{O}_{\mathbf{P}^1}(e_2) \oplus \dots \oplus \mathcal{O}_{\mathbf{P}^1}(e_k)$, a locally free sheaf of rank k on \mathbf{P}^1 , and let $\pi : \mathbf{P}(\mathcal{E}) \rightarrow \mathbf{P}^1$ be the corresponding \mathbf{P}^{k-1} -bundle.

A rational normal scroll X is the image of the map $\iota : \mathbf{P}(\mathcal{E}) \hookrightarrow \mathbf{P}^N := \mathbf{P}H^0(\mathbf{P}(\mathcal{E}), \mathcal{O}_{\mathbf{P}(\mathcal{E})}(1))$. The scroll type of X is defined to be equal to (e_1, e_2, \dots, e_k) .

Remark 2.3. The dimension of X is equal to k , and the degree of X is equal to the degree of \mathcal{E} which is equal to $f := \sum_{i=1}^k e_i$. Moreover, by the Riemann-Roch theorem for vector bundles, $h^0(\mathbf{P}(\mathcal{E}), \mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)) = h^0(\mathbf{P}^1, \mathcal{E}) = \text{rk}(\mathcal{E}) + \deg(\mathcal{E}) = k + \sum_{i=1}^k e_i$, i.e. the dimension of the ambient projective space is equal to $N = k + \sum_{i=1}^k e_i - 1$.

Thus for a rational normal scroll X we obtain $\dim(X) + \deg(X) = k + \sum_{i=1}^k e_i = N + 1$, and consequently a rational normal scroll $X \subseteq \mathbf{P}^N$ is a non-degenerate irreducible variety of minimal degree $f = \text{codim}(X) + 1$.

The scroll X is smooth if and only if all e_i , $i = 1, \dots, k$, are positive. If this is the case, then $\iota : \mathbf{P}(\mathcal{E}) \rightarrow X \subseteq \mathbf{P}^N$ is an isomorphism.

Proposition 2.4. Each linearly normal scroll X over \mathbf{P}^1 is a rational normal scroll.

Proof. If X is a linearly normal scroll over \mathbf{P}^1 , then $X = \iota(\mathbf{P}(\mathcal{E}))$, where $\mathcal{E} = \pi_* \mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$ is a vector bundle over \mathbf{P}^1 and $\iota : \mathbf{P}(\mathcal{E}) \hookrightarrow \mathbf{P}(H^0(\mathcal{E}))$. By Grothendieck's splitting theorem (cf. [4]) every vector bundle over \mathbf{P}^1 splits, i.e. \mathcal{E} is of the form $\mathcal{E} = \oplus_i \mathcal{O}_{\mathbf{P}^1}(e_i)$. \square

A more geometric description of a rational normal scroll is given by the following definition (cf. [6]):

Definition 2.5. Let e_1, e_2, \dots, e_k be integers with $e_1 \geq e_2 \geq \dots \geq e_k \geq 0$ and $\sum_{i=1}^k e_i \geq 2$. Let for $i = 1, \dots, k$, $\phi_i : \mathbf{P}^1 \rightarrow C_i \subseteq \mathbf{P}^{e_i} \subseteq \mathbf{P}^N$, where $N = \sum_{i=1}^k e_i - k - 1$, parametrize a rational normal curve of degree e_i , such that $\mathbf{P}^{e_1}, \dots, \mathbf{P}^{e_k}$ span the whole \mathbf{P}^N . Then

$$X = \overline{\bigcup_{P \in \mathbf{P}^1} \langle \phi_1(P), \dots, \phi_k(P) \rangle}$$

is a rational normal scroll of dimension k , degree $e_1 + \dots + e_k$ and scroll type (e_1, \dots, e_k) . In other words, each fiber of X is spanned by k points where each of these lies on a different rational normal curve. We call these k rational normal curves C_i directrix curves of the scroll.

2.1. The Picard group of rational normal scrolls. Let $H = [\iota^* \mathcal{O}_{\mathbf{P}^N}(1)]$ denote the hyperplane class, and let $F = [\pi^* \mathcal{O}_{\mathbf{P}^1}(1)]$ be the class of a fiber of $\mathbf{P}(\mathcal{E})$. In the following we will use H and F to denote both the classes and divisors in the respective classes.

The *Picard group* of $\mathbf{P}(\mathcal{E})$ is generated by H and F :

$$\mathrm{Pic}(\mathbf{P}(\mathcal{E})) = \mathbf{Z}[H] \oplus \mathbf{Z}[F].$$

We have the following intersection products:

$$H^k = f = \sum_{i=1}^k e_i, \quad H^{k-1} \cdot F = 1, \quad F^2 = 0.$$

A minimal section of $\mathbf{P}(\mathcal{E})$ is given by $B_0 = H - rF$, where $r \in \mathbf{N}$ is maximal such that $H - rF$ is effective, in other words, $B_0 = H - e_1 F$.

2.2. The $g_2^1(C)$ -scroll and $g_3^1(C)$ -scrolls. By Proposition 2.1 there exists exactly one g_2^1 on C . We set

$$S = \overline{\bigcup_{E \in g_2^1(C)} \mathrm{span}(E)} \subseteq \mathbf{P}^{d-2},$$

where $\mathrm{span}(E)$ denotes the line in \mathbf{P}^{d-2} spanned by the two points in the divisor E . For each $|D|$ which is a $g_3^1(C)$ we set

$$V_{|D|} = \overline{\bigcup_{D' \in |D|} \mathrm{span}(D')} \subseteq \mathbf{P}^{d-2},$$

where $\mathrm{span}(D')$ denotes the plane in \mathbf{P}^{d-2} spanned by the three points in the divisor D' .

Proposition 2.6. *Let $C \subseteq \mathbf{P}^{d-2}$ be a smooth linearly normal curve of degree $d \geq 6$ and genus 2, let S be the $g_2^1(C)$ -scroll, and for a linear system $|D|$ which is a $g_3^1(C)$ let $V_{|D|}$ be the $g_3^1(C)$ -scroll associated to $|D|$. The scrolls S and $V_{|D|}$ are rational normal scrolls.*

Proof. The rationality of S and each $V_{|D|}$ is obvious. For the rest notice that if a scroll X contains a linearly normal curve C , then also X has to be linearly normal: If X was the image of a non-degenerate variety in higher-dimensional projective space under some projection, then C had to be as well. We conclude that since C is linearly normal, S and all $V_{|D|}$ are linearly normal. By Proposition 2.4 we can conclude that S and all $V_{|D|}$ are rational normal scrolls. \square

Note that the dimension of S is equal to $\dim(|K_C|) + \dim(\mathrm{span}(E)) = 2$ and that the dimension of $V_{|D|}$ is equal to $\dim(|D|) + \dim(\mathrm{span}(D')) = 3$. By Proposition 2.6 the scrolls S and $V_{|D|}$ are rational normal scrolls which implies by the observations in Remark 2.3 that we obtain the following degrees:

$$(1) \quad \deg(S) = d - 3, \quad \deg(V_{|D|}) = d - 4.$$

Proposition 2.7. *Let $C \subseteq \mathbf{P}^{d-2}$ be a curve of genus 2 and degree $d \geq 6$, and let \mathcal{E} be a \mathbf{P}^1 -bundle such that the image of the map $\iota : \mathbf{P}(\mathcal{E}) \rightarrow \mathbf{P}^{d-2}$ is the $g_2^1(C)$ -scroll S .*

The class of C on $\mathbf{P}(\mathcal{E})$ is equal to $[C] = 2H - (d - 6)F$.

Proof. Write $[C] = aH + bF$ with $a, b \in \mathbf{Z}$. Since $[C].F = 2$, we obtain $a = 2$, and $[C].H = d$ implies that $d = 2(d - 3) + b$, i.e. $b = 6 - d$. \square

Proposition 2.8. *Let $C \subseteq \mathbf{P}^{d-2}$ be a smooth curve of genus 2 and degree $d \geq 6$, and let \mathcal{E} be a \mathbf{P}^1 -bundle such that the image of $\iota : \mathbf{P}(\mathcal{E}) \rightarrow \mathbf{P}^{d-2}$ is a $g_3^1(C)$ -scroll V such that C does not pass through the (possibly empty) singular locus of V .*

The class of C on $\mathbf{P}(\mathcal{E})$ is equal to $[C] = 3H^2 - 2(d - 6)H.F$.

Proof. Since C is of codimension 2 on $\mathbf{P}(\mathcal{E})$, we can write the class of C on $\mathbf{P}(\mathcal{E})$ as $[C] = aH^2 + bH.F$ with $a, b \in \mathbf{Z}$. Since $[C].F = 3$, we obtain $a = 3$, and $[C].H = d$ implies that $d = 3(d - 4) + b$, i.e. $b = 2(6 - d)$. \square

Theorem 5.1 states that I_C is generated by the union of I_S and the ideal of one $g_3^1(C)$ -scroll $V_{|D|}$ that obviously does not contain S . Since the ideal of each rational normal scroll is generated by quadrics, in order to give this statement a sense we have to prove that the ideal I_C is generated by quadrics as well:

Theorem 2.9. *Let C be a smooth curve of genus 2 embedded in \mathbf{P}^{d-2} with a complete linear system of degree $d \geq 6$. The ideal I_C is generated by quadrics.*

Proof. This is Theorem (4.a.1) in [1]. \square

Corollary 2.10. *Let $C \subseteq \mathbf{P}^{d-2}$ be a smooth curve of genus 2 embedded with a complete linear system of degree $d \geq 6$. Then C has no trisecant lines.*

Proof. Since the ideal of C is generated by quadrics, we can write $C = Q_1 \cap \dots \cap Q_r$ where the Q_i are quadrics and $r = h^0(\mathcal{I}_C(2))$. Any line that intersects C in three points, intersects each quadric Q_i in at least three points, consequently it is contained in each Q_i and hence in the intersection of all Q_i 's which is equal to C . \square

3. SCROLL TYPES OF THE $g_2^1(C)$ -SCROLL S

In this section we will list all possible types of the $g_2^1(C)$ -scroll S and give a connection between the complete linear system $|H_C|$ that embeds the curve C into projective space and the scroll type of S . In this way we also give a proof of existence in all cases.

First we give a relation between the degrees of the directrix curves of the $g_2^1(C)$ -scroll:

Proposition 3.1. *Let $C \subseteq \mathbf{P}^{d-2}$ be a smooth curve of degree $d \geq 6$ and genus 2. For the scroll type (e_1, e_2) of the $g_2^1(C)$ -scroll S we have $e_1 - e_2 \leq 3$.*

Proof. Let $C_0 = H - e_1F$ be a minimal section as described as B_0 in general in Section 2.1. Since C and C_0 are effective and C is smooth, so $C_0 \not\subseteq C$, we have $[C].C_0 \geq 0$, which means by Proposition 2.7 that $(2H - (d - 6)F).(H - e_1F) \geq 0$, consequently $2e_1 + 2e_2 - 2e_1 - (d - 6) \geq 0$. Since $d = e_1 + e_2 + 3$ the result follows. \square

Now we will describe the relation between $|H_C|$ with respect to $|K_C|$ and the scroll type of the $g_2^1(C)$ -scroll S . By Equation (1) the degree of S is equal to $d - 3$.

3.1. The case when the degree d of the curve C is even. If the degree d of the embedded curve $C \subseteq \mathbf{P}^{d-2}$ is even, then by Proposition 3.1 the $g_2^1(C)$ -scroll S has scroll type $(\frac{d-2}{2}, \frac{d-4}{2})$ or $(\frac{d}{2}, \frac{d-6}{2})$. The linear system $|H_C - \frac{d-2}{2}K_C|$ is of degree 2 and thus non-empty by the Riemann-Roch theorem for curves and either equal to $|K_C|$ or equal to $|P + Q|$, where P and Q are points on C such that $P + Q \notin |K_C|$.

There is the following relation between $|H_C - \frac{d-2}{2}K_C|$ and the scroll type of S :

Proposition 3.2. *The scroll type of the $g_2^1(C)$ -scroll S is equal to $(\frac{d-2}{2}, \frac{d-4}{2})$ if and only if $|H_C - \frac{d-2}{2}K_C| = |P + Q|$, where P and Q are points on C such that $P + Q \notin |K_C|$.*

Proof. If the scroll type of S is equal to $(\frac{d-2}{2}, \frac{d-4}{2})$, then a minimal section C_0 is of degree $\frac{d-4}{2}$, and a general hyperplane section of S containing C_0 consists of C_0 and $\frac{d-2}{2}$ fibers of S . Consequently, $|H_C| = |\frac{d-2}{2}K_C + P + Q|$, where P and Q are points in $C_0 \cap C$ and $P + Q \notin |K_C|$.

Conversely, if S is of scroll type $(\frac{d}{2}, \frac{d-6}{2})$, a general hyperplane section of S that contains a minimal section C_0 , which is of degree $\frac{d-6}{2}$, decomposes into C_0 and $\frac{d}{2}$ fibers of S . Hence $|H_C| = \frac{d}{2}|K_C|$. \square

3.2. The case when the degree d of the curve C is odd. If the degree d of the embedded curve C is odd, then by Proposition 3.1 the $g_2^1(C)$ -scroll S is of scroll type $(\frac{d-3}{2}, \frac{d-3}{2})$ or $(\frac{d-1}{2}, \frac{d-5}{2})$.

The linear system $|H_C - \frac{d-3}{2}K_C|$ has degree 3, and it is thus non-empty by the Riemann-Roch theorem for curves. It is either equal to $|P + Q + R|$, where P , Q and R are points on C such that none of $P + Q$, $P + R$ and $Q + R$ is a divisor in $|K_C|$, or equal to $|K_C + P|$, where P is a point on C .

The following proposition states the relation between $|H_C - \frac{d-3}{2}K_C|$ and the scroll type of S :

Proposition 3.3. *The $g_2^1(C)$ -scroll S has scroll type $(\frac{d-3}{2}, \frac{d-3}{2})$ if and only if $|H_C - \frac{d-3}{2}K_C| = |P + Q + R|$, where P , Q and R are points on C such that none of $P + Q$, $P + R$ and $Q + R$ is a divisor in $|K_C|$.*

Proof. If S has scroll type $(\frac{d-3}{2}, \frac{d-3}{2})$, then a general hyperplane section of S containing a minimal section C_0 is equal to the union of C_0 and $\frac{d-3}{2}$ fibers of S . We obtain that $|H_C| = |\frac{d-3}{2}K_C + P + Q + R|$, where P, Q, R are points lying on $C_0 \cap C$ and none of $P + Q$, $P + R$ or $Q + R$ is a divisor in $|K_C|$.

Conversely, if the scroll type of S is equal to $(\frac{d-1}{2}, \frac{d-5}{2})$, then a general hyperplane section of S that contains C_0 decomposes into a minimal section C_0 and $\frac{d-1}{2}$ fibers of S . Consequently, $|H_C| = |\frac{d-1}{2}K_C + P|$ where P is a point in $C_0 \cap C$. \square

4. SCROLL TYPES OF $g_3^1(C)$ -SCROLLS $V_{|D|}$

In this section we will first give a relation between the degrees of the three directrix curves of a scroll $V_{|D|}$, and then we will give a connection between $|H_C|$ and the scroll type of $V_{|D|}$ for a given $|D| \in G_3^1(C)$.

Proposition 4.1. *If V is a $g_3^1(C)$ -scroll such that the curve C does not intersect the (possibly empty) singular locus of V , then for its scroll type (e_1, e_2, e_3) we have $2e_1 - e_2 - e_3 \leq 4$.*

Proof. Let $B_0 = H - e_1F$ denote a minimal section of the bundle $\mathbf{P}(\mathcal{E})$. Since we have $h^0(\mathcal{O}_V(H - B_0)) = h^0(\mathcal{O}_V(e_1F)) = e_1 + 1 \geq 1$, B_0 is contained in at least one hyperplane, consequently B_0 does not span all of \mathbf{P}^{d-2} . Since C spans all of \mathbf{P}^{d-2} , B_0 cannot contain C , thus we have that $[C].B_0 \geq 0$, i.e. we know that $(3H^2 - 2(d-6)HF).(H - e_1F) \geq 0$ by Proposition 2.8, which means that $3e_1 + 3e_2 + 3e_3 - 3e_1 - 2(d-6) \geq 0$. The result follows from $d = e_1 + e_2 + e_3 + 4$. \square

Proposition 4.2. *If $V = V_{|D|}$ is a singular scroll of scroll type $(e_1, e_2, 0)$ such that the curve C intersects the singular locus of V , then e_1 and e_2 satisfy the following: $e_1 - e_2 \leq 3$.*

Proof. If $V = V_{|D|}$ is a singular $g_3^1(C)$ -scroll of type $(e_1, e_2, 0)$ such that C intersects its singular locus, then a point $P \in \text{sing}(V) \cap C$ is a basepoint of $|D|$, i.e. $|D| = |K_C + P|$. The projection from P maps C to a curve C' of degree $d - 1$ in \mathbf{P}^{d-3} , and it maps $V_{|D|}$ to the $g_2^1(C')$ -scroll of type (e_1, e_2) . By Proposition 3.1 we obtain $e_1 - e_2 \leq 3$. \square

We will now come to the converse of Proposition 4.2, i.e. the existence part:

Proposition 4.3. *If e_1 and e_2 are integers with $e_1 \geq e_2 \geq 0$, $e_1 - e_2 \leq 3$ and $e_1 + e_2 = d - 4$ with $d \geq 6$, then there exists a curve C of genus 2 and a divisor class $|H_C|$ on C of degree d that embeds C into \mathbf{P}^{d-2} such that there exists a $g_3^1(C)$ -scroll of type $(e_1, e_2, 0)$ such that its singular locus intersects the curve C .*

Proof. Let $e_1 \geq e_2 \geq 0$ be integers with $e_1 - e_2 \leq 3$ and $e_1 + e_2 = d - 4$. By the results in Section 3 there exists a curve C of genus 2, embedded with a system $|H'|$ of degree $d - 1$ into \mathbf{P}^{d-3} such that its $g_2^1(C)$ -scroll is of type (e_1, e_2) . Take a point P on C and reembed the curve C with the linear system $|H_C| := |H' + P|$ into \mathbf{P}^{d-2} . The cone over the $g_2^1(C)$ -scroll in \mathbf{P}^{d-3} with P as vertex is a $g_3^1(C)$ -scroll in \mathbf{P}^{d-2} of type $(e_1, e_2, 0)$, and the point P lies in the intersection of its singular locus and the curve C . \square

Now we will describe all scroll types a $g_3^1(C)$ -scroll $V_{|D|}$ can have as $|D|$ varies. We will distinguish between $|D|$ with one basepoint and $|D|$ basepoint-free.

4.1. The case when $V_{|D|}$ is smooth or $V_{|D|}$ is singular and C does not pass through the singular locus of $V_{|D|}$. If $V_{|D|}$ is smooth or $V_{|D|}$ is singular, but the curve C does not pass through the singular locus of $V_{|D|}$, then $|D|$ necessarily has to be basepoint-free, since a basepoint of $|D|$ is a point in the intersection of C with the singular locus of $V_{|D|}$.

In order to describe the possible scroll types with a relation to $|H_C|$ we use a formula given

in [5]. We give an alternative proof of the fact that the following numbers d_i determine the scroll type of a $g_3^1(C)$ -scroll $V_{|D|}$:

Proposition 4.4. ([5], p. 114) *Given a basepoint-free $|D| \in G_3^1(C)$, set*

$$\begin{aligned} d_0 &= h^0(\mathcal{O}_C(H_C)) - h^0(\mathcal{O}_C(H_C - D)), \\ d_1 &= h^0(\mathcal{O}_C(H_C - D)) - h^0(\mathcal{O}_C(H_C - 2D)), \\ d_2 &= h^0(\mathcal{O}_C(H_C - 2D)) - h^0(\mathcal{O}_C(H_C - 3D)), \\ &\vdots \\ d_{\lfloor \frac{d}{3} \rfloor} &= h^0(\mathcal{O}_C(H_C - \lfloor \frac{d}{3} \rfloor D)) - \underbrace{h^0(\mathcal{O}_C(H_C - (\lfloor \frac{d}{3} \rfloor + 1)D))}_{=0}. \end{aligned}$$

The scroll type (e_1, e_2, e_3) of $V_{|D|}$ is then given by

$$\begin{aligned} e_1 &= \#\{j | d_j \geq 1\} - 1, \\ e_2 &= \#\{j | d_j \geq 2\} - 1, \\ e_3 &= \#\{j | d_j \geq 3\} - 1. \end{aligned}$$

Proof. Since C and V are both linearly normal and span all of \mathbf{P}^{d-2} , there is an isomorphism $H^0(C, \mathcal{O}_C(H_C)) \cong H^0(\mathbf{P}(\mathcal{E}), \mathcal{O}_{\mathbf{P}(\mathcal{E})}(H)) \cong H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(e_1) \oplus \mathcal{O}_{\mathbf{P}^1}(e_2) \oplus \mathcal{O}_{\mathbf{P}^1}(e_3))$, and thus we obtain

$$H^0(C, \mathcal{O}_C(H - iD)) \cong H^0(\mathbf{P}(\mathcal{E}), \mathcal{O}_{\mathbf{P}(\mathcal{E})}(H - iF)) \cong H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(e_1 - i) \oplus \mathcal{O}_{\mathbf{P}^1}(e_2 - i) \oplus \mathcal{O}_{\mathbf{P}^1}(e_3 - i))$$

for all $i \in \mathbf{N}_0$.

Consequently we obtain

$$d_i = \begin{cases} 3, & \text{if } 0 \leq i \leq e_3, \\ 2, & \text{if } e_3 + 1 \leq i \leq e_2, \\ 1, & \text{if } e_2 + 1 \leq i \leq e_1. \end{cases}$$

□

Using the formula in Proposition 4.4 we will now give all possible scroll types of $V_{|D|}$, in relation to $|H_C|$:

d	Conditions on $ H_C $	Scroll type of $V_{ D }$
$d \equiv_{(3)} 0$	$ H_C - \frac{d-3}{3}D = D $	$(\frac{d}{3}, \frac{d}{3} - 2, \frac{d}{3} - 2)$
	$ H_C - \frac{d-3}{3}D \neq D $	$(\frac{d}{3} - 1, \frac{d}{3} - 1, \frac{d}{3} - 2)$
$d \equiv_{(3)} 1$	$ H_C - \frac{d-1}{3}D \neq \emptyset$	$(\frac{d-1}{3}, \frac{d-1}{3} - 1, \frac{d-1}{3} - 2)$
	$ H_C - \frac{d-1}{3}D = \emptyset$	$(\frac{d-1}{3} - 1, \frac{d-1}{3} - 1, \frac{d-1}{3} - 1)$
$d \equiv_{(3)} 2$	$ H_C - \frac{d-2}{3}D = K_C $	$(\frac{d-2}{3}, \frac{d-2}{3}, \frac{d-2}{3} - 2)$
	$ H_C - \frac{d-2}{3}D = P + Q ,$ $P + Q \notin K_C $	$(\frac{d-2}{3}, \frac{d-2}{3} - 1, \frac{d-2}{3} - 1)$

4.2. The case when $V_{|D|}$ is singular and C passes through the singular locus of $V_{|D|}$. If $V_{|D|}$ is singular, then its scroll type is equal to $(e_1, e_2, 0)$ where $e_1 \geq e_2 \geq 0$. By Proposition 4.2, if C passes through the singular locus of $V_{|D|}$, then $e_2 = 0$ is only possible for $d \leq 7$. For $d \geq 8$ the scroll $V_{|D|}$ has exactly one singular point which we will denote by P . For $d = 6$ and $d = 7$ let P denote one point in the singular locus of $V_{|D|}$. If the system $|D|$ has a basepoint P , then the curve passes through the singular locus of the scroll $V_{|D|}$. As suggested in Proposition 4.3, given $|H_C|$ and $|D| = |K_C + P|$, we can find the scroll type of $V_{|D|}$ by projecting from the point P and using Proposition 3.2 and Proposition 3.3 for the $g_2^1(C')$ -scroll, where C' is the image of C under the projection. Let in this situation P' always denote the point in $|K_C - P|$.

4.2.1. The case when the degree d of the curve C is even. If d is even, then, since each linear system of degree 2 is non-empty by the Riemann-Roch theorem for curves, we can write $|H_C| = |\frac{d}{2}K_C| = |\frac{d-2}{2}K_C + P + P'|$ or $|H_C| = |\frac{d-2}{2}K_C + Q_1 + Q_2| = |\frac{d-4}{2}K_C + Q_1 + Q_2 + P + P'|$, where Q_1 and Q_2 are points on C such that $Q_1 + Q_2 \notin |K_C|$.

Proposition 4.5. *Given $|D| = |K_C + P|$, the scroll type of $V_{|D|}$ is equal to $(\frac{d-4}{2}, \frac{d-4}{2}, 0)$ if and only if $|H_C - \frac{d-2}{2}K_C - P| = \emptyset$.*

Proof. Let $|H_C - \frac{d-2}{2}K_C - P| = \emptyset$. Projecting C from P yields a curve C' of genus 2 and degree $d-1$ which is embedded in \mathbf{P}^{d-3} with the linear system $|H'| := |H_C - P|$. Under this projection the scroll $V_{|D|}$ maps to the $g_2^1(C')$ -scroll S' . The scroll $V_{|D|}$ is thus the cone over S' with P as vertex, so if S' is of scroll type (e_1, e_2) , then $V_{|D|}$ is of scroll type $(e_1, e_2, 0)$. Since $|H' - \frac{d-2}{2}K_C| = \emptyset$, we obtain by Proposition 3.3 that the scroll type of $V_{|D|}$ is equal to $(\frac{d-4}{2}, \frac{d-4}{2}, 0)$.

Conversely, if $|H_C - \frac{d-2}{2}K_C - P| \neq \emptyset$, then the same procedure as above, i.e. projecting from P , yields $|H' - \frac{d-2}{2}K_C| \neq \emptyset$, i.e. by Proposition 3.3 the scroll type of $V_{|D|}$ is equal to $(\frac{d-2}{2}, \frac{d-6}{2}, 0)$. \square

4.2.2. *The case when the degree d of the curve C is odd.* If d is odd, then we can write either $|H_C| = |\frac{d-1}{2}K_C + Q| = |\frac{d-3}{2}K_C + Q + P + P'|$ or $|H_C| = |\frac{d-3}{2}K_C + \sum_{i=1}^3 Q_i| = |\frac{d-5}{2}K_C + \sum_{i=1}^3 Q_i + P + P'|$, where Q_1, Q_2 and Q_3 are points on C such that $Q_i + Q_j \notin |K_C|$ for all $i, j \in \{1, 2, 3\}, i \neq j$.

Proposition 4.6. *The scroll type of the $g_3^1(C)$ -scroll $V_{|D|}$ associated to $|D| = |K_C + P|$ is equal to $(\frac{d-3}{2}, \frac{d-5}{2}, 0)$ if and only if $|H_C - \frac{d-1}{2}K_C - P| = \emptyset$.*

Proof. We use the same idea as in the proof of Proposition 4.5. Let $|H_C - \frac{d-1}{2}K_C - P| = \emptyset$. Projecting C from P yields a curve C' of genus 2 in \mathbf{P}^{d-3} , embedded with the system $|H'| := |H_C - P|$. Since $|H' - \frac{d-1}{2}K_C| = \emptyset$, by Proposition 3.2 the scroll type of $V_{|D|}$ is equal to $(\frac{d-3}{2}, \frac{d-5}{2}, 0)$.

Conversely, if $|H_C - \frac{d-1}{2}K_C - P| \neq \emptyset$, then $|H' - \frac{d-1}{2}K_C| \neq \emptyset$. By Proposition 3.3 we obtain that the scroll type of $V_{|D|}$ is equal to $(\frac{d-1}{2}, \frac{d-7}{2}, 0)$. \square

Example 4.7. *In order to illustrate Propositions 4.4 and 4.6 we study the case $d = 7$, i.e. let now $C \subseteq \mathbf{P}^5$ be a smooth curve of genus 2 and degree 7. Below we list all scroll types of a $g_3^1(C)$ -scroll $V_{|D|}$ in relation to $|H_C|$. In our description we will consider the systems $|H_C - 3K_C|$ and $|H_C - 2D|$. Both are of degree 1, and thus each of these can either be empty or consist of one point.*

Conditions on the basepoint locus of $ D $	$ H_C $	$ H_C - 2D $	$ H_C - 3K_C $	Scroll type of $V_{ D }$
No basepoints	$ D + 2K_C $	\emptyset	\emptyset	$(1, 1, 1)$
No basepoints	$ D + K_C + Q_1 + Q_2 $, $Q_1 + Q_2 \notin K_C $, $ D - Q_1 - Q_2 = \emptyset$, $ D - Q'_1 - Q'_2 = \emptyset$, where $ Q'_1 = K_C - Q_1 $, $ Q'_2 = K_C - Q_2 $	\emptyset	\emptyset	$(1, 1, 1)$
No basepoints	$ D + K_C + Q_1 + Q_2 $, $Q_1 + Q_2 \notin K_C $, $ D - Q_1 - Q_2 = \emptyset$, $ D - Q'_1 - Q'_2 \neq \emptyset$, where $ Q'_1 = K_C - Q_1 $, $ Q'_2 = K_C - Q_2 $	\emptyset	$ D - Q'_1 - Q'_2 $ (non-empty)	$(1, 1, 1)$

Conditions on the basepoint locus of $ D $	$ H_C $	$ H_C - 2D $	$ H_C - 3K_C $	Scroll type of $V_{ D }$
No basepoints	$ D + K_C + Q_1 + Q_2 $, $Q_1 + Q_2 \notin K_C $, $ D - Q_1 - Q_2 \neq \emptyset$, $ D - Q'_1 - Q'_2 = \emptyset$, where $ Q'_1 = K_C - Q_1 $, $ Q'_2 = K_C - Q_2 $	non-empty	\emptyset	$(2, 1, 0)$
No basepoints	$ D + K_C + Q_1 + Q_2 $, $Q_1 + Q_2 \notin K_C $, $ D - Q_1 - Q_2 \neq \emptyset$, $ D - Q'_1 - Q'_2 \neq \emptyset$, where $ Q'_1 = K_C - Q_1 $, $ Q'_2 = K_C - Q_2 $	non-empty	$ D - Q'_1 - Q'_2 $ (non-empty)	$(2, 1, 0)$
One basepoint P	$ D + K_C + Q_1 + Q_2 $ $= 2K_C + P + Q_1 + Q_2 $, $Q_1 + Q_2 \notin K_C $, $P \neq Q_i$ for all $i \in \{1, 2\}$, $P + Q_i \notin K_C $ for all $i \in \{1, 2\}$	\emptyset	\emptyset	$(2, 1, 0)$
One basepoint P	$ D + K_C + Q_1 + Q_2 $ $= 2K_C + P + Q_1 + Q_2 $, $Q_1 + Q_2 \notin K_C $, $P \neq Q_i$ for all $i \in \{1, 2\}$, $P + Q_i \in K_C $ for some $i \in \{1, 2\}$	\emptyset	non-empty, different from P	$(2, 1, 0)$
One basepoint P	$ 2D + Q $ $= 2K_C + 2P + Q $, $P + Q \notin K_C $, $2P \notin K_C $	Q (non-empty)	\emptyset	$(2, 1, 0)$
One basepoint P	$ 2D + Q $ $= 2K_C + 2P + Q $, $P + Q \notin K_C $, $P \neq Q$, $2P \in K_C $	Q (non-empty)	Q (non-empty, different from P)	$(2, 1, 0)$
One basepoint P	$ 3K_C + P $	$ K_C - P $ (non-empty)	P	$(3, 0, 0)$

5. THE IDEAL OF C AS A SUM OF SCROLLAR IDEALS

In this section we will show that the ideal I_C of a linearly normal embedded curve $C \subseteq \mathbf{P}^{d-2}$ of genus 2 and degree $d \geq 6$ is generated by the ideals I_S and I_V , where S is the $g_2^1(C)$ -scroll and $V = V_{|D|}$ is a $g_3^1(C)$ -scroll not containing S . In other words, we will prove the following main theorem in this section:

Theorem 5.1. *Let C be a smooth and irreducible curve of genus 2, linearly normal embedded in \mathbf{P}^{d-2} with a complete linear system $|H_C|$ of degree $d \geq 6$. Moreover, let S be the $g_2^1(C)$ -scroll, and let V be a $g_3^1(C)$ -scroll that does not contain S . Then we have*

$$I_S + I_V = I_C.$$

We see that in this section we are only interested in $g_3^1(C)$ -scrolls $V_{|D|}$ that do not contain the $g_2^1(C)$ -scroll S . For this purpose we will now give a criterion for when a given $g_3^1(C)$ -scroll $V_{|D|}$ does not contain the $g_2^1(C)$ -scroll S :

Proposition 5.2. *Let $C \subseteq \mathbf{P}^{d-2}$ be a curve of genus 2 and degree $d \geq 6$, embedded with the system $|H_C|$, and let S be the $g_2^1(C)$ -scroll. A $g_3^1(C)$ -scroll $V = V_{|D|}$ contains S if and only if at least one of the following holds:*

- $|D|$ has a basepoint,
- $d = 6$ and $|H_C - D|$ has a basepoint or
- $d = 7$ and $|H_C| = |D + 2K_C|$.

Proof. If $|D|$ has a basepoint P , then $|D| = |K_C + P|$, hence each fiber of S is contained in a fiber of $V_{|D|}$, and consequently $V_{|D|}$ contains S .

Conversely, if $S \subseteq V_{|D|}$ and $|D|$ is basepoint-free, then each fiber of $V_{|D|}$ intersects each fiber of S in one point, since if it did not, then each fiber of S had to be contained in a fiber of V which meant that $|D|$ had a basepoint. This implies that each fiber of $V_{|D|}$, which is a plane, intersects the scroll S in a directrix curve of S . This curve is a smooth rational planar curve, consequently the degree of this curve is equal to 1 or 2. This means that, since the degree of C is greater or equal to 6, the scroll type of S is equal to $(2, 1)$ or $(2, 2)$, i.e. $d = 6$ or $d = 7$. Since each fiber of $V_{|D|}$ intersects each fiber of S in one point, every divisor in $|D + K_C|$ spans a \mathbf{P}^3 and thus $h^0(H - (D + K_C)) = d - 5$, which implies that in the case $d = 6$ we obtain that $|H - D - K_C|$ contains one point, and that in the case $d = 7$, $|H - D - K_C| = |K_C|$. \square

Before we will prove Theorem 5.1 we show that $S \cap V = C$ for any $g_3^1(C)$ -scroll V that does not contain S :

Proposition 5.3. *Let $C \subseteq \mathbf{P}^{d-2}$ be a smooth and irreducible linearly normal curve of genus 2 and degree $d \geq 6$. For a $g_3^1(C)$ -scroll $V = V_{|D|}$ that does not contain the $g_2^1(C)$ -scroll S the following holds:*

$$S \cap V = C.$$

Proof. Obviously, $C \subseteq S \cap V$. In the case $d = 6$ the claim follows by Bézout's Theorem, since then $S \cap V$ is of degree 6 and dimension 1. Let now $d \geq 7$. If $S \cap V$ is more than C , then it must at least contain one line: If $S \cap V \supseteq C \cup P$ for a point P that does not lie on C , then P lies on one fiber F_0 of the scroll S . But since P does not lie on the curve, each quadric that contains V intersects F_0 in at least three points, consequently the whole fiber F_0 must be contained in each quadric that contains V , and since the ideal I_V is generated by quadrics, F_0 is contained in $S \cap V$. Now there are a priori two possibilities for a fiber F of S to be contained in V :

- (1) F is contained in one of the fibers of $V = V_{|D|}$; this implies that the system $|D|$ has a basepoint, $|D| = |K_C + P|$, and consequently $S \subseteq V$.
- (2) F is intersecting each fiber of V in one point. Since F is a fiber of S , the point of intersection lies on C for exactly two fibers of V . Projecting C from F yields a curve C' of genus 2 and degree $d - 2$, linearly normal embedded in \mathbf{P}^{d-4} with the linear system $|H_C - K_C|$. The curve C' lies on the scrollar surface S' which is the image of V under the projection from F . A general fiber of V is projected to a fiber in S' , and the three points in the intersection of C with a general fiber in V are projected to three points on a fiber in S' , which is impossible by Corollary 2.10, unless C was a curve of degree 7 in \mathbf{P}^5 . If C is a curve of degree 7 that projects to a curve C' of degree 5 on $\mathbf{P}^1 \times \mathbf{P}^1$, then $|H_C| = |D + 2K_C|$, and the $g_3^1(C)$ -scroll $V = V_{|D|}$ contains the $g_2^1(C)$ -scroll S by Proposition 5.2.

This proves that the intersection $S \cap V$ cannot contain any line, i.e. in total we obtain $S \cap V = C$. \square

Proof of Theorem 5.1:

Let S be the $g_2^1(C)$ -scroll, and let $V = V_{|D|}$ be a $g_3^1(C)$ -scroll that does not contain S . There is the following short exact sequence of ideal sheaves:

$$0 \rightarrow \mathcal{I}_{S \cup V} \rightarrow \mathcal{I}_V \rightarrow \mathcal{I}_{S \cap V}|_S \rightarrow 0.$$

By Proposition 5.3 we have $S \cap V = C$, and moreover we know that $\mathcal{I}_C|_S = \mathcal{O}_S(-C)$. We thus obtain the following short exact sequence:

$$0 \rightarrow \mathcal{I}_{S \cup V} \rightarrow \mathcal{I}_V \rightarrow \mathcal{O}_S(-C) \rightarrow 0.$$

Tensoring with $\mathcal{O}_{\mathbf{P}^{d-2}}(2H)$ and restricting yields the following exact sequence:

$$0 \rightarrow \mathcal{I}_{S \cup V}(2H) \rightarrow \mathcal{I}_V(2H) \rightarrow \mathcal{O}_S(2H - C) \rightarrow 0.$$

Taking the long exact sequence in cohomology yields

$$0 \rightarrow H^0(\mathcal{I}_{S \cup V}(2)) \rightarrow H^0(\mathcal{I}_V(2)) \rightarrow H^0(\mathcal{O}_S(2H - C)) \rightarrow H^1(\mathcal{I}_{S \cup V}(2)) \rightarrow 0.$$

Note that $h^1(\mathcal{I}_V(2)) = 0$ since V is projectively normal.

Since $[C] = 2H - (d - 6)F$ on S , we can write the above sequence in the following form:

$$0 \rightarrow H^0(\mathcal{I}_{S \cup V}(2)) \rightarrow H^0(\mathcal{I}_V(2)) \xrightarrow{\psi} H^0(\mathcal{O}_S((d-6)F)) \rightarrow H^1(\mathcal{I}_{S \cup V}(2)) \rightarrow 0.$$

Our aim is now to show the following claim:

Claim 5.4. *For each $|D| \in G_3^1(C) = \{g_3^1(C)'s\}$ such that $V_{|D|}$ does not contain S , the map $\psi : H^0(\mathcal{I}_V(2)) \rightarrow H^0(\mathcal{O}_S((d-6)F))$ defined via*

$$\psi(Q) := \begin{cases} 0 & \text{if } S \subseteq Q, \\ Q \cap S - C \in |(d-6)F| & \text{if } S \not\subseteq Q \end{cases}$$

is surjective.

If the claim is true, then we have $h^1(\mathcal{I}_{S \cup V}(2)) = 0$, and thus the short exact sequence

$$0 \rightarrow \mathcal{I}_{S \cup V}(2) \rightarrow \mathcal{I}_S(2) \oplus \mathcal{I}_V(2) \rightarrow \underbrace{\mathcal{I}_{S \cap V}(2)}_{=C} \rightarrow 0$$

gives the following long exact sequence in cohomology:

$$0 \rightarrow H^0(\mathcal{I}_{S \cup V}(2)) \rightarrow H^0(\mathcal{I}_S(2)) \oplus H^0(\mathcal{I}_V(2)) \rightarrow H^0(\mathcal{I}_C(2)) \rightarrow 0.$$

This implies that

$$\begin{aligned} h^0(\mathcal{I}_C(2)) &= \dim(H^0(\mathcal{I}_S(2)) \oplus H^0(\mathcal{I}_V(2))) - h^0(\mathcal{I}_{S \cup V}(2)) \\ &= \dim(H^0(\mathcal{I}_S(2)) + H^0(\mathcal{I}_V(2))). \end{aligned}$$

This argument implies that, since $H^0(\mathcal{I}_S(2)) + H^0(\mathcal{I}_V(2)) \subseteq H^0(\mathcal{I}_C(2))$,

$$H^0(\mathcal{I}_S(2)) + H^0(\mathcal{I}_V(2)) = H^0(\mathcal{I}_C(2)),$$

but since all I_S , I_V and I_C are generated by quadrics, we obtain $I_S + I_V = I_C$.

Proof of Claim 5.4:

Now we will prove by induction that the map

$$\psi : H^0(\mathcal{I}_V(2)) \rightarrow H^0(\mathcal{O}_S((d-6)F))$$

as defined above is surjective:

The induction start: $d = 6$ and $d = 7$:

For $d = 6$ the surjectivity of ψ is obvious. More precisely, if $C \subseteq \mathbf{P}^4$ is a curve of degree 6, and if $|D|$ is a basepoint-free $g_3^1(C)$ such that $|H_C - D|$ is basepoint-free as well, then by Proposition 5.2 the scroll $V_{|D|} =: Q_6$ is a quadric that does not contain the $g_2^1(C)$ -scroll S .

For a curve $C \subseteq \mathbf{P}^5$ of degree 7 let $V_{|D|}$ be a $g_3^1(C)$ -scroll that does not contain the $g_2^1(C)$ -scroll S . For any two quadrics $Q_1 \neq Q_2$ in \mathbf{P}^5 their intersection $Q_1 \cap Q_2$ is a complete

intersection of dimension 3 and degree 4, hence if Q_1 and Q_2 both contained S and V , then we have $Q_1 \cap Q_2 = V \cup \mathbf{P}^3$. Since $S \subseteq Q_1 \cap Q_2$ and S is irreducible, we must have $S \subseteq V$ or $S \subseteq \mathbf{P}^3$, but since S spans all of \mathbf{P}^5 and by hypothesis S is not contained in V , both cases are impossible.

This shows that $h^0(\mathcal{I}_{S \cup V}(2)) \leq 1$, and consequently we obtain

$$\dim(H^0(\mathcal{I}_S(2)) + H^0(\mathcal{I}_V(2))) \geq h^0(\mathcal{I}_S(2)) + h^0(\mathcal{I}_V(2)) - 1 = 8 = h^0(\mathcal{I}_C(2)),$$

and thus ψ is surjective.

The induction step: $d \geq 8$:

Pick $d - 8$ fibers F_1, \dots, F_{d-8} on S . Let R_1 and R_2 be two points on C such that $R_1 + R_2$ is not a divisor in $|K_C|$ and such that $|R_i| \neq |H_C - D - 2K_C|$, $i = 1, 2$, in the case if $d = 8$ and the linear system $|H_C - D - 2K_C|$ is non-empty. Moreover, let R'_1 and R'_2 be two points on C such that $R_1 + R'_1$ and $R_2 + R'_2$ are divisors in $|K_C|$. Projecting C from the line L_R spanned by R_1 and R_2 yields a curve C' of degree $d - 2$ in \mathbf{P}^{d-4} , embedded with the system $|H_C - R_1 - R_2|$. Under this projection the $g_2^1(C)$ -scroll S maps to the $g_2^1(C')$ -scroll S' , and the scroll $V_{|D|}$ maps to a $g_3^1(C')$ -scroll $V'_{|D|}$ that does not contain S' . Notice that the choice of R_1 and R_2 ensures that $V'_{|D|}$ does not contain S' also in the cases $d = 8$, $|H_C - D - 2K| \neq \emptyset$ and $d = 9$, $|H_C| = |D + 3K_C|$. (In general it would not have been clear that the system $|H_C - R_1 - R_2|$ does not belong to the cases given in Proposition 5.2.)

By the induction hypothesis we find a quadric $Q_{d-2} \subseteq \mathbf{P}^{d-4}$ which contains $V'_{|D|}$ but not S' , and which contains the fibers F'_1, \dots, F'_{d-8} , where F'_i , $i = 1, \dots, d - 8$, denotes the image of F_i under the projection. The cone over Q_{d-2} with the line L_R as vertex is then a quadric Q_d in \mathbf{P}^{d-2} which contains $V_{|D|}$ and not S . Moreover, Q_d contains the fibers F_1, \dots, F_{d-8} and two more fibers of S : The fiber F_{R_1} spanned by R_1 and R'_1 intersects the quadric Q_d in three points, counted with multiplicity: The quadric Q_d intersects this line in at least the two points R_1 and R'_1 , and since the quadric is singular along the line L_R , the quadric Q_d intersects F_{R_1} in the point R_1 with at least multiplicity 2. Consequently Q_d must contain the fiber F_{R_1} . The same argument applies to the fiber F_{R_2} which is spanned by the points R_2 and R'_2 .

By degree reasons $Q_d \cap S$ cannot contain more than C , F_{R_1} , F_{R_2} and F_1, \dots, F_{d-8} . Consequently, $\psi(Q_d) = F_{R_1} \cup F_{R_2} \cup F_1 \cup \dots \cup F_{d-8}$. Since the divisors $F_{R_1} + F_{R_2} + F_1 + \dots + F_{d-8}$, where R_1 and R_2 run through all points on C and F_1, \dots, F_{d-8} run through all fibers of S , span the linear system $|(d - 6)F|$, the linearity of ψ together with varying the points R_1 and R_2 and the fibers F_1, \dots, F_{d-8} yields the surjectivity of ψ . \square

Acknowledgements. This paper arose from parts of my Ph.D. thesis. I wish to thank my advisor Kristian Ranestad for many interesting discussions and very helpful feedback.

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